LITERATURE CITED

1. N. I. Nikitenko, The Net Method in Research on Nonstationary Heat and Mass Transfer [in Russian], Naukova Dumka, Kiev (1971).
2. K. M. Magomedov and A. S. Kholodov, "Construction of difference schemes for hyperbolic equations via characteristic relationships," Zh. Vychisl. Mat. Mat. Fiz., 9, No. 2 (1969).
3. V. V. Nekrasov, "The characteristics method in numerical solution of multidimensional boundary-value problems," Zh. Vychisl. Mat. Mat. Fiz., 15, No. 6 (1975).
4. N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov, Equations in Partial Derivatives in Mathematical Physics [in Russian], Vysshaya Shkola, Moscow (1970).
5. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems, Wiley (1967),

COUPLED DYNAMIC THERMOELASTICITY PROBLEM FOR A HALF SPACE WITH THERMAL "MEMORY"

V. L. Kolpashchikov and S. Yu. Yanovskii

UDC 539.3

The coupled dynamic thermoelasticity problem is solved for a half space endowed with thermal "memory." The properties of the generated thermoelastic waves are discussed.

Chen and Gurtin [1], elaborating the general nonlinear theory of conduction of Gurtin and Pipkin [2], extended it to include strain in the medium. They derived nonlinear functional defining relations for the thermoviscoelasticity of bodies with thermal and strain memory, whereby the prior history of the variation of the thermodynamic and mechanical characteristics is taken into consideration:

$$\psi(X,\ t)=\Psi(\Lambda^t),\ \ \sigma(X,\ t)=\Sigma(\Lambda^t),\ \ \eta(X,\ t)=N(\Lambda^t),\ q(X,\ t)=Q(\Lambda^t), \tag{1}$$

where $\Lambda^t = (F,\ T,\ \overline{F}^t,\ \overline{T}^t,\ \overline{g}^t)$ is the thermal history of the process (1).

In the present article, we investigate the one-dimensional coupled dynamic problem for a linear thermoelastic isotropic half space. After linearization of the system of defining relations (1) with regard for the laws of conservation of momentum and energy, we arrive at the following dimensionless system of equations for the temperature, stress, and displacement fields induced in the half space:

$$\begin{gathered}
M^2\frac{\partial^2\theta}{\partial\tau^2}+\frac{\partial\theta}{\partial\tau}+\int_0^\infty \beta'(s)\,\frac{\partial\theta(x,\ \tau-s)}{\partial\tau}\,ds=\frac{\partial^2\theta}{\partial x^2}+\int_0^\infty \alpha'(s)\,\frac{\partial^2\theta(x,\ \tau-s)}{\partial x^2}-\varepsilon\,\frac{\partial^3 u}{\partial x\partial\tau^2},\\
\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial\tau^2}=\frac{\partial\theta}{\partial x}-\Gamma\int_0^\infty \gamma(s)\,\frac{\partial\theta(x,\ \tau-s)}{\partial x}\,ds,\\
\sigma_x=\frac{\partial u}{\partial x}-\theta+\Gamma\int_0^\infty \gamma(s)\,\theta(x,\ \tau-s)\,ds,\\
\sigma_y=\sigma_z=\frac{\varkappa_4}{2\varkappa_3+\varkappa_4}\,\sigma_x-\frac{2\varkappa_3}{2\varkappa_3+\varkappa_4}\left[\theta-\Gamma\int_0^\infty \gamma(s)\,\theta(x,\ \tau-s)\,ds\right],
\end{gathered} \tag{2}$$

where

$$\varkappa_1\delta_{ij}=D_F E(\Lambda_0),\ \ -\varkappa_2\delta_{ij}=D_\theta\Sigma(\Lambda_0),\ \ \varkappa_3(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk})+\varkappa_4\delta_{ij}\delta_{kl}=D_F\Sigma(\Lambda_0).$$

V. I. Lenin Belorussian State University. A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 36, No. 6, pp. 1093-1099, June, 1979. Original article submitted June 29, 1978.

 0022-0841/79/3606-0720$07.50

At the boundary of the half space ($\xi = 0$), the stresses are equal to zero, and the temperature changes at the initial time to a value T_m, thereafter remaining constant. The temperature, stress, and their time derivatives are equal to zero at the initial time. The temperature and stresses are bounded at infinity:

$$\theta(x,\ 0) = \sigma_x(x,\ 0) = \left.\frac{\partial\theta(x,\ \tau)}{\partial\tau}\right|_{\tau=0} = \left.\frac{\partial\sigma_x(x,\ \tau)}{\partial\tau}\right|_{\tau=0} = 0, \tag{3}$$

$$\sigma_x(0,\ \tau) = 0,\quad \theta(0,\ \tau) = H(\tau), \tag{4}$$

$$\lim_{x\to\infty} \sigma_x(x,\ \tau) < \infty,\quad \lim_{x\to\infty} \theta(x,\ \tau) < \infty. \tag{5}$$

Applying the Laplace transform with respect to the variable τ to (2), (4), and (5), subject to the initial conditions (3), we obtain

$$\frac{d^2\bar{\theta}(x,\ p)}{dx^2} - \frac{p}{\bar{\alpha}}\,[M^2 + \bar{\beta} + \varepsilon(1 - \Gamma\bar{\gamma})]\,\bar{\theta}(x,\ p) = \frac{\varepsilon p}{\bar{\alpha}}\,\bar{\sigma}_x(x,\ p); \tag{6}$$

$$\frac{d^2\bar{\sigma}_x(x,\ p)}{dx^2} - p^2\bar{\sigma}_x(x,\ p) = p^2(1 - \Gamma\bar{\gamma})\,\bar{\theta}(x,\ p), \tag{7}$$

$$\bar{\sigma}_y = \bar{\sigma}_z = \frac{\varkappa_4}{2\varkappa_3 + \varkappa_4}\,\bar{\sigma}_x - \frac{2\varkappa_3}{2\varkappa_3 + \varkappa_4}\,(1 - \Gamma\bar{\gamma})\,\bar{\theta}, \tag{8}$$

$$\bar{\sigma}_x(0,\ p) = 0,\quad \bar{\theta}(0,\ p) = \frac{1}{p}, \tag{9}$$

$$\lim_{x\to\infty} \bar{\sigma}_x(x,\ p) < \infty,\quad \lim_{x\to\infty} \bar{\theta}(x,\ p) < \infty. \tag{10}$$

Solving the boundary-value problem (6)-(10), we obtain expressions for the transforms of the temperature and stresses:

$$\bar{\theta}(x,\ p) = \frac{(p^2 - \gamma_2^2)\exp(-\gamma_2 x) - (p^2 - \gamma_1^2)\exp(-\gamma_1 x)}{p(\gamma_1^2 - \gamma_2^2)}, \tag{11}$$

$$\bar{\sigma}_x(x,\ p) = \frac{p(1 - \Gamma\bar{\gamma})}{\gamma_1^2 - \gamma_2^2}\,[\exp(-\gamma_1 x) - \exp(-\gamma_2 x)], \tag{12}$$

where

$$\gamma_{1,2}^2 = \frac{p}{2}\left\{p + \frac{M^2 + \bar{\beta}}{\bar{\alpha}} + \frac{\varepsilon}{\bar{\alpha}}\,(1 - \Gamma\bar{\gamma}) \pm \sqrt{\left[p + \frac{M^2 + \bar{\beta}}{\bar{\alpha}} + \frac{\varepsilon}{\bar{\alpha}}\,(1 - \Gamma\bar{\gamma})\right]^2 - 4p\,\frac{M^2 + \bar{\beta}}{\bar{\alpha}}}\right\}. \tag{13}$$

In order to revert from the transforms to the inverse transforms in (11) and (12), it is necessary to specify the relaxation functions. We write the relaxation functions for the heat flux and internal energy in the form [3]

$$\alpha(\tau) = 1 + A\tau + o(\tau^2),\quad \beta(\tau) = 1 + B\tau + o(\tau^2). \tag{14}$$

We specify the temperature relaxation function for the stresses in the form

$$\gamma(\tau) = 1 + \Gamma_1\tau + o(\tau^2). \tag{15}$$

We analyze the asymptotic cases of small and large times. In the neighborhood of the point $p = \infty$, we obtain from (13)

$$\gamma_{1,2}^2 \approx \frac{p}{2}\left\{p(1 + M^2 + \varepsilon \pm \sqrt{K}) + \left[1 - AM^2 - \varepsilon(\Gamma + A) \pm \frac{L}{\sqrt{K}}\right] + o\left(\frac{1}{p}\right)\right\}, \tag{16}$$

$$\gamma_{1,2} \approx \frac{p}{v_{1,2}} + \delta_{1,2} + o\left(\frac{1}{p}\right), \tag{17}$$

where

$$K = (1 + \varepsilon + M^2)^2 - 4M^2, \tag{18}$$

$$L = (1 + M^2 + \varepsilon)\,[1 - AM^2 - \varepsilon(A + \Gamma)] - 2(1 - AM^2), \tag{19}$$

$$v_{1,2} = \sqrt{2}/\sqrt{1 + M^2 + \varepsilon \pm \sqrt{K}}, \tag{20}$$

$$\delta_{1,2} = \frac{v_{1,2}}{4} \cdot \left[1 - AM^2 - \varepsilon(\Gamma + A) \pm \frac{L}{\sqrt{K}}\right]. \tag{21}$$

Making use of expressions (16)-(21), we obtain an approximate solution in the vicinity of the wave fronts for small times:

$$\theta(x, \tau) \approx \frac{1}{2\sqrt{K}} \left\{\exp(-\delta_2 x)\, H\left(\tau - \frac{x}{v_2}\right)\left[1 - M^2 - \varepsilon + \sqrt{K} - \right.\right.$$
$$- \left[1 - AM^2 - \varepsilon(\Gamma + A) - \frac{L}{\sqrt{K}} + \frac{L}{K}\,(1 - M^2 - \varepsilon + \sqrt{K})\right] \times$$
$$\times \left.\left(\tau - \frac{x}{v_2}\right)\right] - \exp(-\delta_1 x)\, H\left(\tau - \frac{x}{v_1}\right)\left[1 - M^2 - \varepsilon - \sqrt{K} - \right.$$
$$\left.\left. - \left[1 - AM^2 - \varepsilon(\Gamma + A) + \frac{L}{\sqrt{K}} + \frac{L}{K}(1 - M^2 - \varepsilon - \sqrt{K})\right]\left(\tau - \frac{x}{v_1}\right)\right]\right\}, \tag{22}$$

$$\sigma_x(x, \tau) \approx \frac{1}{\sqrt{K}} \left\{\exp(-\delta_1 x)\, H\left(\tau - \frac{x}{v_1}\right)\left[1 - \left(\Gamma + \frac{L}{K}\right)\left(\tau - \frac{x}{v_1}\right)\right]\right.$$
$$\left. - \exp(-\delta_2 x)\, H\left(\tau - \frac{x}{v_2}\right)\left[1 - \left(\Gamma + \frac{L}{K}\right)\left(\tau - \frac{x}{v_2}\right)\right]\right\}.$$

If the relaxation function for the heat flux is specified in the form of a Maxwell–Cattaneo kernel:

$$\alpha(\tau) = \frac{M^2}{M_*^2} \exp\left[-\frac{\tau}{M_*^2}\right], \tag{24}$$

and if temperature relaxation of the stresses and relaxation of the internal energy do not take place:

$$\gamma(\tau) = 0, \quad \beta(\tau) = 0, \quad \beta(0) = c_v/\tau_r, \tag{25}$$

then, substituting (24) and (25) into (11)-(13) and specifying the linearization coefficients as

$$\varkappa_1 = \alpha_t T_0 (3\lambda + 2\mu), \quad \varkappa_2 = \alpha_t (3\lambda + 2\mu), \quad \varkappa_3 = \mu, \quad \varkappa_4 = \lambda, \tag{26}$$

we arrive at the generalized thermomechanics solution [4].

We now consider the other extreme case of large times. We adopt the relaxation functions in the form [5]

$$\alpha(\tau) = \exp(-\omega_1\tau), \quad \beta(\tau) = \exp(-\omega_2\tau), \quad \gamma(\tau) = \exp(-\omega_3\tau). \tag{27}$$

In the limit $p \to 0$, we infer from (13)

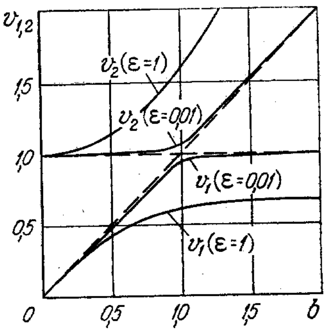
$$\gamma_1^2 \approx p\, \frac{\omega_1}{\omega_2\omega_3}\, [\omega_3(M^2\omega_2 + 1) + \varepsilon\omega_2(\omega_3 - \Gamma)], \quad \gamma_2^2 \approx 0.$$

It is evident from (11), (12), and (28) that the large-time solution of problem (2)-(5) satisfies the classical equations of coupled thermoelasticity and exactly coincides with the large-time solution of the classical problem, in which case

$$\omega_1 = \frac{1}{M^2}, \quad \omega_2 = \infty, \quad \omega_3 = \infty. \tag{29}$$

Analysis of the Solution. 1. It is evident from the solutions (22) and (23) that two temperature waves and two normal-stress waves propagate in an elastic half space with velocities v_1 and v_2. By analogy with generalized thermomechanics, we call them fast (v_2) and slow (v_1) waves. The nature of the dependence of the velocities on the dimensionless heat propagation velocity is illustrated in Fig. 1 for two values of the coupling parameter. The dashed curves represent the velocities for the uncoupled problem. As $b \to 0$, the velocity of the slow wave tends to zero (as b), while the fast-wave velocity tends to unity; as $b \to \infty$, the fast-wave velocity grows without bound as $b(1 + \varepsilon)^{1/2}$, while the slow-wave velocity tends to $(1 + \varepsilon)^{-1/2}$. The analogous dependences of generalized thermomechanics are given in [6,7].

2. If for comparison with the results of generalized thermomechanics we specify the



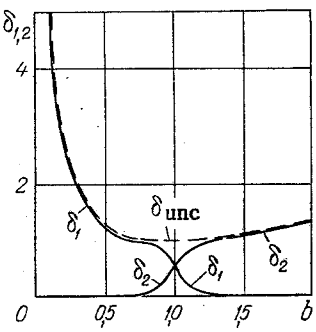


Fig. 1 Fig. 2

Fig. 1. Velocities of fast (v_2) and slow (v_1) waves versus heat-propagation velocity for two values of the coupling parameter (v_1, v_2, b, and ε are dimensionless).

Fig. 2. Decay rates of fast (δ_2) and slow (δ_1) waves versus heat-propagation velocity for a heat-flux relaxation parameter A = −1, stress–temperature relaxation parameter Γ = 0, and coupling parameter ε = 0.01 (δ_1, δ_2, δ_{unc}, b, and ε are dimensionless).

linearization coefficients in the form (26) and take the heat-propagation velocity equal to the velocity of second sound, we find that thermal memory effects cause the wave velocities of the coupled problem to increase. Thus, for aluminum $M_* = 2.157$ [8]

$$\frac{v_1^*}{v_1} \approx 0.81, \quad \frac{v_2^*}{v_2} \approx 0.99, \quad (\varepsilon_* = 1). \tag{30}$$

For small coupling parameters

$$\frac{v_1^*}{v_1} \to \frac{w^*}{w} = \frac{M}{M_*}, \quad \frac{v_2^*}{v_2} \to \frac{v^*}{v}. \tag{31}$$

3. At the wave fronts, the temperature and stress suffer discontinuities:

$$[\theta]_{\tau=\frac{x}{v_{1,2}}} = \frac{1}{2\sqrt{K}} (\sqrt{K} \pm \varepsilon \pm M^2 \mp 1) \exp(-\delta_{1,2} x), \tag{32}$$

$$[\sigma]_{\tau=\frac{x}{v_{1,2}}} = \pm \frac{1}{\sqrt{K}} \exp(-\delta_{1,2} x), \tag{33}$$

which decay into the depth of the half space.

4. Thermal memory effects strongly affect the nature of the decay processes. It follows from the results of Chen and Nunziato [9] that $A \le 0$. Additional investigations are required to determine the sign of the coefficient Γ.

The dependence of the decay rate of the fast (δ_2) and slow (δ_1) waves on b is given in Fig. 2 for A = −1 and Γ = 0. The dashed curve represents the decay rate of the thermal wave in the uncoupled problem. For small values of b (i.e., for media having a small density and large heat capacity), the decay rate of the slow wave increases without bound, asymptotically approaching the decay rate of the uncoupled problem, while the fast-wave decay rate tends to zero. As $b \to \infty$, the fast-wave decay rate increases without bound, asymptotically approaching the decay rate of the uncoupled problem, while the slow-wave decay rate tends to the finite limit $\varepsilon/_2[1 - (\Gamma + A)(1 + \varepsilon)]/(1 + \varepsilon)^{3/2}$. In the neighborhood of the point b = 1, a "resonance effect" is observed, the stress discontinuities at both wave fronts decay at the same rate [for ε = 0, with a decay rate $(1 - A)/4$].

5. Ignoring the coupling of thermal and mechanical effects (ε = 0) and assuming the absence of temperature relaxation of the stresses (Γ = 0), from (22) and (23) we deduce a solution of the uncoupled thermoelasticity problem [11] for small times in the vicinity of the wave fronts:

$$\theta(x, \tau) \approx H(\tau - xM) \exp\left[-\frac{1 - AM^2}{2M} x\right],$$

$$\sigma_x(x,\ \tau) \approx \frac{1}{1-M^2}\left\{H(\tau-x)\left[1-\frac{1-AM^2}{M^2-1}\ (\tau-x)\right]-\right. \tag{34}$$

$$\left.-\exp\left[-\frac{1-AM^2}{2M}\,x\right]H(\tau-xM)\left[1-\frac{1-AM^2}{M^2-1}\ (\tau-xM)\right]\right\}. \tag{35}$$

We note that the presence of temperature relaxation of the stresses does not influence the behavior of magnitude of the discontinuities of the uncoupled problem.

6. If the relaxation function for the heat flux is specified in the form of a Maxwell–Cattaneo kernel, if relaxation of the internal energy and temperature relaxation of the stresses do not occur, and if the linearization coefficients are specified in the form (26), then a solution of the coupled thermoelasticity problem of generalized thermomechanics can be obtained from the general solution for the transforms (11)-(13).

7. The large-time solution of the problem satisfies the classical thermoelasticity equations and exactly coincides with the large-time solution of the classical problem with the relaxation functions specified in the form (27) and (29).

NOTATION

ξ, coordinate normal to the surface of the half space; t, time; T , temperature of the natural state of the half space; $\tilde{\alpha}(t)$, relaxation function for the heat flux; $\tilde{\beta}(t)$, relaxation function for internal energy; $\tilde{\gamma}(t)$, temperature relaxation function for stresses; $c_V = D_\theta E(\Lambda_0)$, volume specific heat; λ_t, thermal conductivity; a, thermal diffusivity; μ, λ, Lamé coefficients; ρ, density of material; α_t, coefficient of linear thermal expansion; E, elastic modulus; τ_r, heat-flux relaxation time; u_ξ, displacements normal to surface of half space; $\sigma_{\xi\xi}$, $\sigma_{\chi\chi}$, $\sigma_{\mu\mu}$, normal stresses; $v = [(2\varkappa_3 + \varkappa_4)/\rho]^{1/2}$, elastic-wave velocity for media with thermal memory; $w = (\alpha(0)/c_V)^{1/2}$, velocity of thermal disturbances for media with thermal memory; $v^* = ((2\mu + \lambda)/\rho)^{1/2}$, velocity of longitudinal elastic waves; $w^* = (a/\tau_r)^{1/2}$, velocity of thermal disturbances in generalized thermomechanics; $\Lambda_0 = (0, T_0, T_0 i, 0)$, equilibrium thermal history; ν, Poisson ratio; T, instantaneous temperature; ψ, free energy; η, entropy; q, heat flux; $F = \partial u_\xi/\partial\xi$, strain; $g = \partial T/\partial\xi$, temperature gradient; $\overline{F}^t$, $\overline{T}^t$, $\overline{g}^t$, total histories; $H(\tau) = \begin{cases}1 & \tau \geqslant 0\\ 0 & \tau < 0\end{cases}$. Dimensionless quantities: $\tau = t\beta(0)v^2/\alpha(0)$, $x = \xi\beta(0)\ v/\alpha(0)$, $u = u_\xi\beta(0)\ v(2\varkappa_3 + \varkappa_4)/\alpha(0)(T_m - T_0)\varkappa_2$; $\theta = (T - T_0)/(T_m - T_0)$, $\alpha(\tau) = \tilde{\alpha}(t)/\alpha(0)$, $\beta(\tau) = \tilde{\beta}(t)/\beta(0)$, $\gamma(\tau) = \tilde{\gamma}(t)/\gamma(0)$; $\sigma_i = \sigma_{jj}/\varkappa_2(T_m - T_0)$, $M^2 = v^2/w^2$, $b^2 = 1/M^2$, $\varepsilon = \varkappa_1\varkappa_2 v^2/(2\varkappa_3 + \varkappa_4)\ \alpha(0)$; $A = \alpha'(0)/\beta(0)\ v^2$, $B = \beta'(0)\ \alpha(0)/\beta^2(0)\ v^2$, $\Gamma = \gamma(0)\ \alpha(0)/\varkappa_2\beta(0)\ v^2$; $\Gamma_1 = \gamma'(0)\ \alpha(0)/\gamma(0)\ \beta(0)\ v^2$, $M_*^2 = v_*^2/w_*^2$, $b_*^2 = 1/M_*^2$, $\varepsilon_* = \alpha_t^2 T_0(3\lambda + 2\mu)^2/(2\mu + \lambda)\ c_v$.

LITERATURE CITED

1. P. J. Chen and M. E. Gurtin, Z. Angew. Math. Phys., 21, 232 (1970).
2. M. E. Gurtin and A. C. Pipkin, Arch. Rat. Mech. Anal., 31, 113 (1968).
3. D. E. Amos and P. J. Chen, Trans. ASME, Ser. E: J. Basic Eng., 92, 244 (1970).
4. F. R. Norwood and W. E. Warren, Q. J. Mech. Appl. Math., 22, No. 3, 283 (1969).
5. F. R. Norwood, Trans. ASME, Ser. E: J. Basic Eng., 94, 3, 35 (1972).
6. E. B. Popov, Prikl. Mat. Mekh., 31, No. 2, 328 (1967).
7. J. D. Achenbach, J. Mech. Phys. Solids, 16, No. 4, 273 (1968).
8. Ya. S. Podstrigach and Yu. M. Kolyano, Generalized Thermomechanics [in Russian], Naukova Dumka, Kiev (1976).
9. P. J. Chen and J. W. Nunziato, Z. Angew. Math. Phys., 25, 791 (1974).
10. A. V. Lykov, Heat and Mass Transfer [in Russian], Énergiya, Moscow (1971).
11. V. L. Kolpashchikov and S. Yu. Yanovskii, in: Aspects of the Theory of Transport Processes [in Russian], ITMO, Minsk (1977), p. 124.